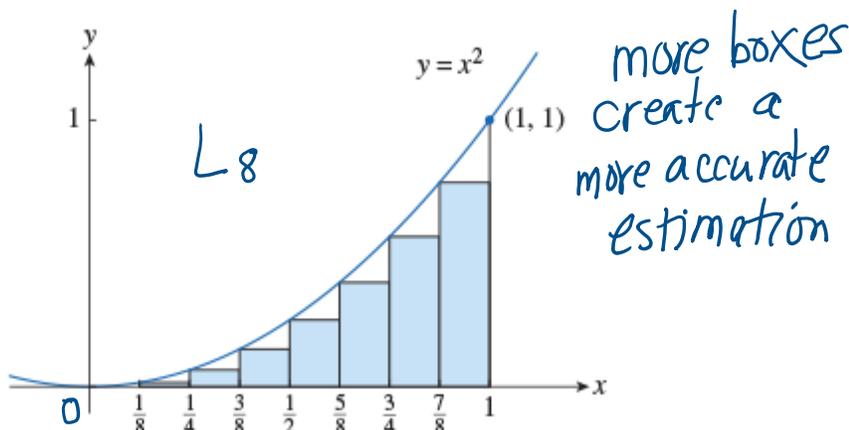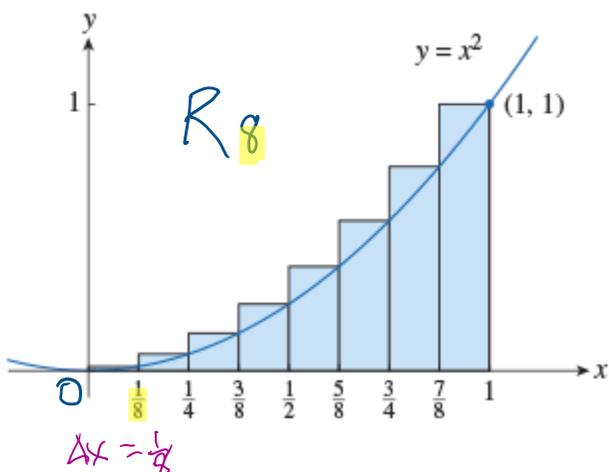REVISIT: set up R_6 under $f(x) = \sin x$ from $x = 0$ to $x = \dfrac{3\pi}{2}$ using *sigma notation*.

x_0 $\quad$ x_1 $\quad$ x_2 $\quad$ x_3 $\quad$ x_4 $\quad$ x_5 $\quad$ x_6 ← n

Δx

pattern

$\dfrac{0\pi}{4}$ $\quad$ $\dfrac{1\pi}{4}$ $\quad$ $\dfrac{\pi}{2} = \dfrac{2\pi}{4}$ $\quad$ $\dfrac{3\pi}{4}$ $\quad$ $\dfrac{4\pi}{4}$ $\quad$ $\dfrac{5\pi}{4}$ $\quad$ $\dfrac{3\pi}{2} = \dfrac{6\pi}{4}$ $\Rightarrow$ $\dfrac{i\pi}{4}$
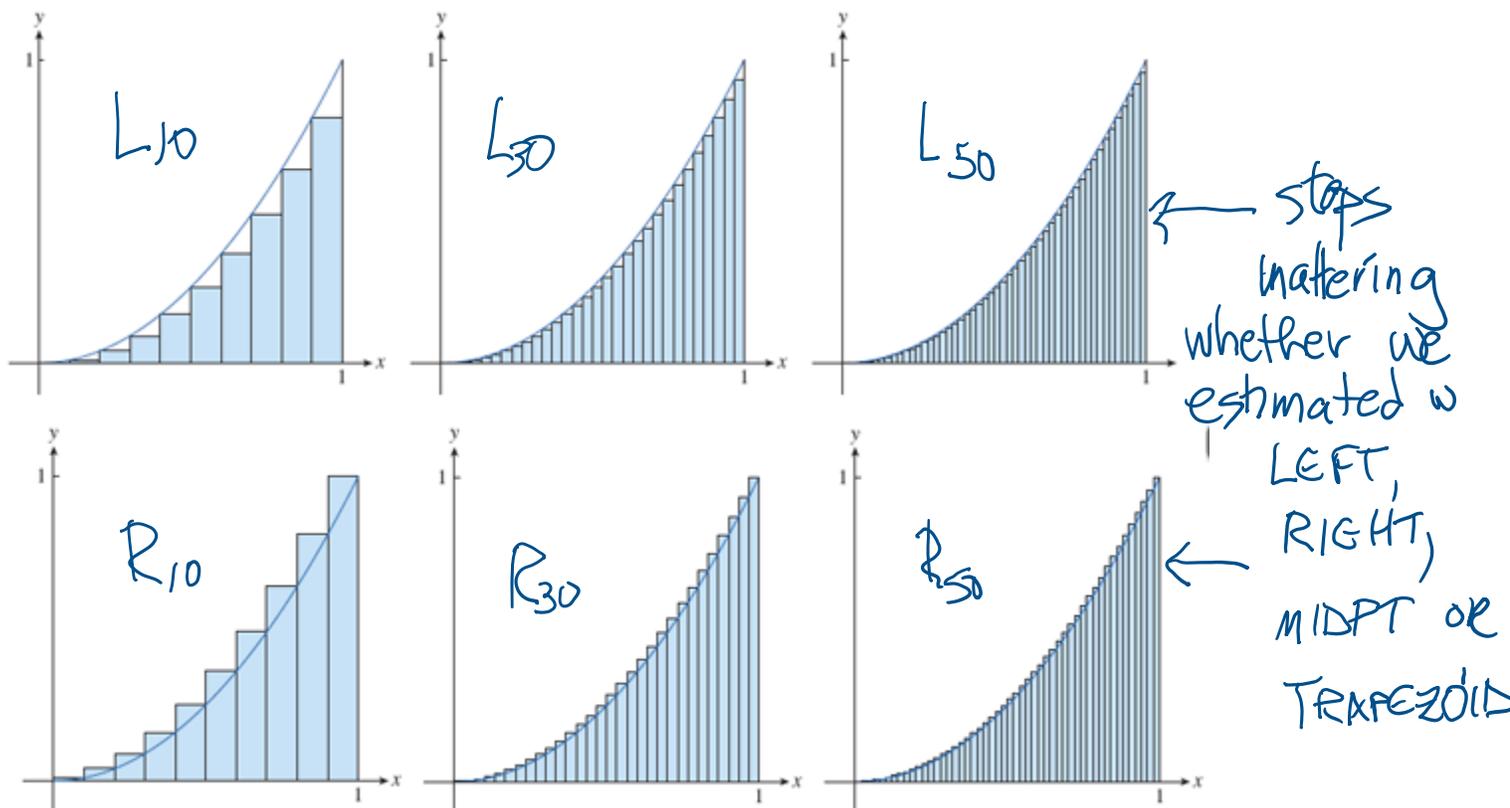
$$R_6 = \frac{\pi}{4}\left(\sum_{i=1}^{6} \sin\left(\frac{i\pi}{4}\right)\right)$$

$$L_6 = \frac{\pi}{4}\left(\sum_{i=0}^{5} \sin\left(\frac{i\pi}{4}\right)\right)$$

L_4 : $y = x^2$, $(1,1)$; $\Delta x = 1/4$

R_4 : $y = x^2$, $(1,1)$

R_8 : $y = x^2$, $(1,1)$; $\Delta x = \frac{1}{8}$

L_8 : $y = x^2$, $(1,1)$

more boxes create a more accurate estimation
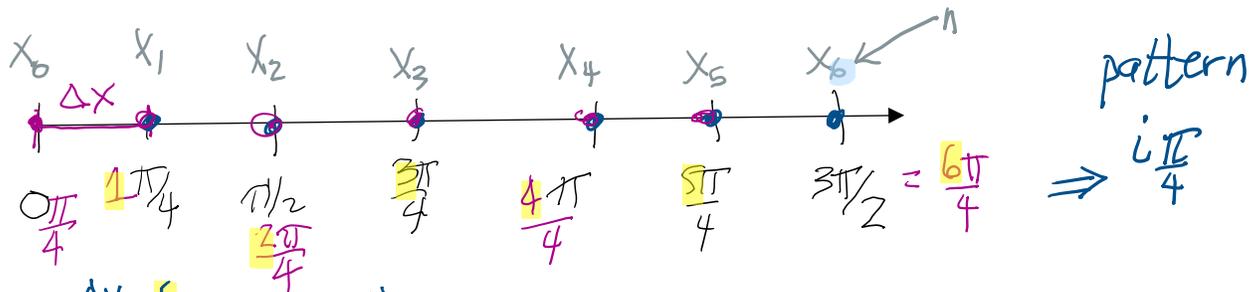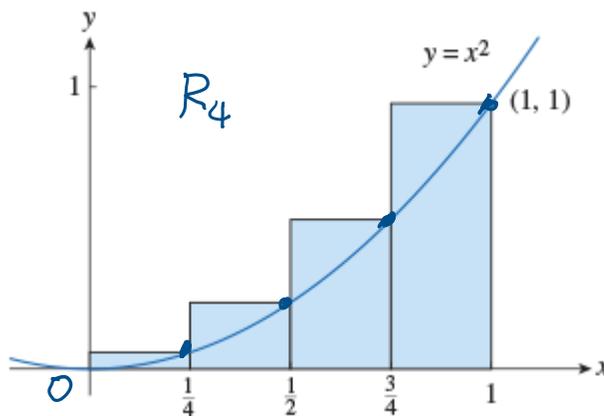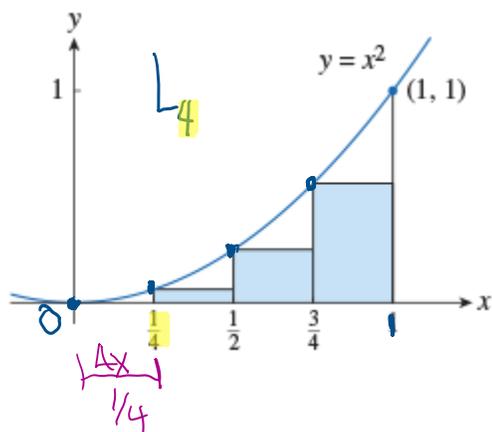
Remember, to increase the accuracy of the estimate, _increase the number of boxes used (n)_

L_{10}

L_{30}

L_{50}

← stops mattering whether we estimated w LEFT, RIGHT, MIDPT or TRAPEZOID

R_{10}

R_{30}

R_{50}

What is long term solution to find the EXACT area under the curve???

use INFINITELY MANY boxes !   ☺

First, expand to "n" slices:
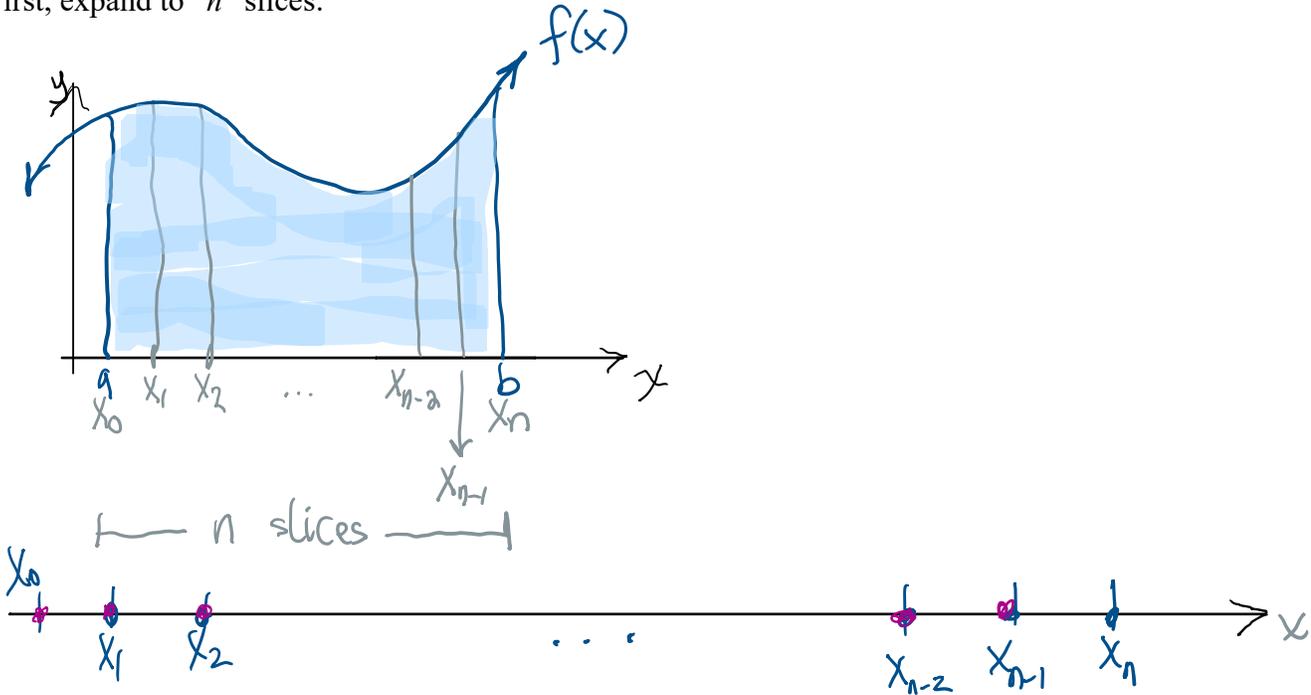
$$R_n = \Delta x \left(f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n) \right)$$

$$= \boxed{\Delta x \sum_{i=1}^{n} f(x_i)} \leftarrow$$

$$L_n = \Delta x \left(f(x_0) + f(x_1) + \ldots + f(x_{n-2}) + f(x_{n-1}) \right)$$

$$b(h_1 + h_2 + h_3)$$
$$= bh_1 + bh_2 + bh_3$$

Now, use infinitely many slices:
$$A = \lim_{n \to \infty} \Delta x \sum_{i=1}^{n} f(x_i)$$

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x}$$

Examples using the above definition of area:

ex. Determine the function and its interval whose area is equal to:
$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{2}{n} \left(5 + \frac{2i}{n} \right)^{10}$$

$\overset{\Delta x}{\downarrow} \quad \overset{f(x_i)}{\overbrace{\qquad}}$

$$x_i = \textcircled{a} + \Delta x \cdot i$$
$$\boxed{5 + \frac{2}{n} \cdot i}$$

$[a, b]$

$$\Delta x = \frac{b-a}{n}$$
$$\frac{2}{n} = \frac{b-5}{n} \Rightarrow 2 = b - 5$$
$$\boxed{b = 7}$$

$$\llcorner x_i = 5 + \frac{2i}{n}$$

$$(x_i)^{10} \text{ or } \boxed{f(x) = x^{10}}$$
on
$$\boxed{[5, 7]}$$

ex. Given $f(x) = \cos x$ on $\left[0, \frac{\pi}{2}\right]$ use the definition to express area using sigma and limit notation.

$$\Delta x = \frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$$

$$x_i = a + \Delta x \cdot i$$
$$= 0 + \frac{\pi}{2n} \cdot i = \frac{\pi i}{2n} = x_i$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \overset{\Delta x}{\frac{\pi}{2n}} \cos x_i = \boxed{\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{2n} \cos\left(\frac{\pi i}{2n}\right)}$$

Definite Integral
Section 5.2

If f is defined on $[a,b]$ then $\displaystyle\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$, where
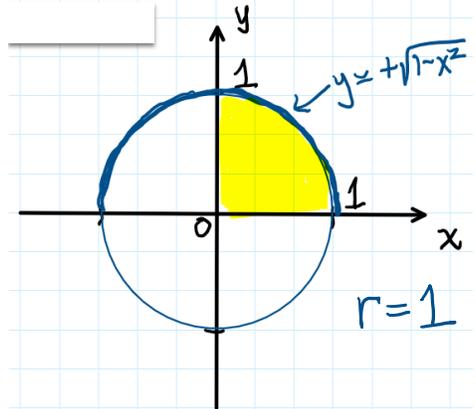
- $\Delta x = \frac{b-a}{n}$
- $x_i = a + \Delta x \cdot i$

upper bound

limits or bounds

lower bound

$$\int_a^b f(x) \, dx \leftarrow \text{defined variable being integrated}$$

function

integrand

↑ integral sign — elongated s (sum)

For now, let's do an example using the shape of a *standard* area:

ex. $\int_0^1 \sqrt{1-x^2}\,dx = A_{\triangle} = \frac{\pi r^2}{4} = \boxed{\frac{\pi}{4}}$

$y = +\sqrt{1-x^2}$

$r = 1$

FTC

Fundamental Theorem of Calculus, Part 1

Establishes a connection between

differential calculus and integral calculus.

derivatives
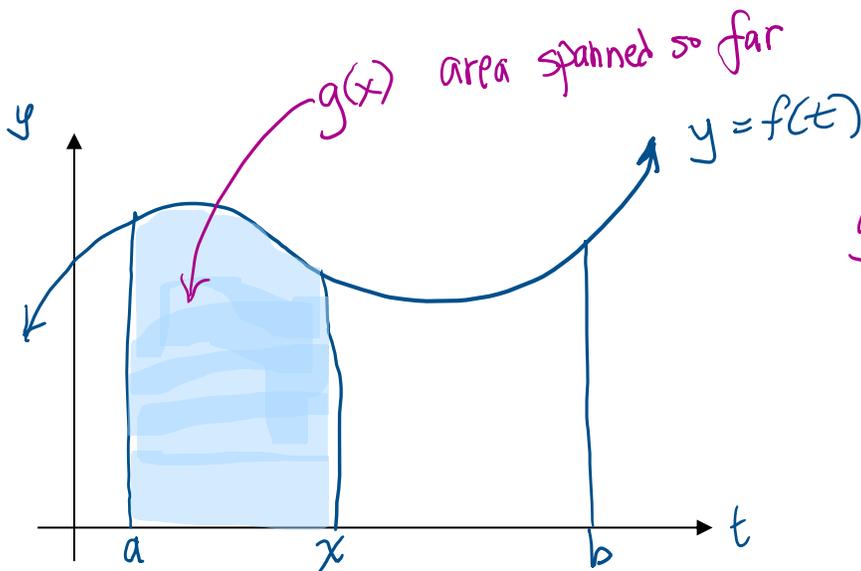
$x^2 \quad \sqrt{x}$

$e^x \quad \ln x$

They are inverse processes:

If f is a **positive** function then

$g(x)$ can be interpreted as

the area under the graph of f from a to x

where x can vary from a to b.

g(x) area spanned so far

$y = f(t)$

$g(x)$ is an antiderivative of f

$\therefore g'(x) = f(x)$ for $a < x < b$

FTC Part 1: $\int_a^x f(t)\, dt = g(x)$ where f is continuous on $[a, b]$

Review Common General Antiderivatives   **Section 4.8**

recall: f is the original function and F is its **general antiderivative**

Power Function: $f(x) = x^n$ $F = \dfrac{x^{n+1}}{n+1} + C$

Exponential Function: $f(x) = e^x$ $F = e^x + C$

Logarithmic Function: $f(x) = \dfrac{1}{x}$ $F = \ln|x| + C$

Trigonometric Functions: $f(x) = \cos x$ $F = \sin x + C$

$f(x) = \sin x$ $F = -\cos x + C$ $(\cos x)' = -\sin x$
$-(\cos x)' = \sin x$

Don't forget to add $+C$

Use integral notation to find antiderivatives or **indefinite integral:**

← no bounds on integral sign

$$F = \frac{x^{n+1}}{n+1} + C$$

ex. $\displaystyle\int x^4 \, dx = \frac{x^5}{5} + C$

ex. $\displaystyle\int 5x^9 \, dx = 5\int x^9 \, dx$

$$= 5 \cdot \frac{x^{10}}{10} + C = \boxed{\frac{x^{10}}{2} + C}$$

ex. $\displaystyle\int \frac{1}{x^5} \, dx = \int x^{-5} \, dx = \frac{x^{-4}}{-4} + C$

$$= \boxed{-\frac{1}{4x^4} + C}$$

$$\tfrac{1}{2} + 1 = \tfrac{3}{2}$$

ex. $\displaystyle\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + C$

$$= \boxed{\frac{2}{3} \cdot x^{3/2} + C}$$

ex. $\displaystyle\int \sec^2 x \, dx = \boxed{\tan x + C}$

$$(\tan x)' = \sec^2 x$$